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ON SONIC FLOWS IMPINGING ON WEDGE-SHAPED OBSTRUCTIONS*

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The problem of plane symmetric sonic flows of ideal (inviscid and non-heat-conducting) gas impinging on wedge-shaped obstructions, which from now on we shall call a wedge, including, in particular, a wedge, a plane, and a wedge-shaped notch, is considered. The characteristic features of the flow in question are: 1) the region G of subsonic flow adajcent to the tip of the wedge is bounded, 2) the sonic lines (SL) forming the boundary of G are rectilinear, and 3) the flow is symmetric about the bisectrix of the outer angle between the axis of the flow and the generator of the wedge. Here and henceforth a line lying within the stream on which the Mach number is $M=1$ and on at least one side of which $M<1$, will be called the sonic line. At least two of the features of the flow in question listed above were known earlier. Thus the fact that the region G is bounded by the straight SL from below along the flow follows from /1, 2/, and the boundedness from above follows from /3/. Moreover, by virtue of the theorem proved in /4/ the SL in the subcritical flow with $M\leqslant 1$ can only be a straight line (see also /5/). The third property, although it was not specially stressed, is also fairly obvious. Therefore, the main result of this section consists of constructing specific cases and determining the function $f(\sigma)\equiv l/h$, where σ is the half-angle of the wedge tip ($0\leqslant\sigma\leqslant\pi$), l is the distance along its generatrix (or the axis of the flow) to the SL, and h is the half-width of the incoming flow. The function $f(\sigma)$ which also depends on the properties of the gas (in the case of a real gas the property is the adiabatic index $\varkappa$) is obtained from the solution of the problem and varies from zero when $\sigma=0$, to infinity as $\sigma\to\pi$.

1. Let a uniform subsonic, or sonic as $x=-\infty$, stream of half-width h, velocity V_∞ and density ρ_∞ impinge on a wedge-shaped body (Fig.1a) with a tip angle of 2σ. In what follows, h will be regarded as the linear scale. The pressure in the space surrounding the stream is p_∞, is constant at its boundary, and either smaller than, or equal to the corresponding critical value. The X axis of the Cartesian XY-coordinates coincides with the axis (or, more accurately, with the plane) of symmetry of the stream and the body, and the origin of coordinates with the tip of the body. When $0<\sigma<\pi/2$, we have flow past a wedge, when $\sigma=\pi/2$ it becomes flow past a plane, and when $\pi/2<\sigma<\pi$, we have the interaction of the flow with a wedge-shaped cavity. In addition to XY, we shall introduce the Cartesian xy-coordinate system obtained by rotating the initial system by an angle $\sigma/2$. The velocity vector V, its modulus V and its projections u and v on the x and y axes, will be referred to the initial velocity. Then, if ϑ is the angle between V and x axis, then the sector $-\sigma/2\leqslant\vartheta\leqslant\sigma/2$ of the circle $0\leqslant V\leqslant V_\infty\leqslant 1$ in the hodograph plane (Fig.1b) will correspond to the upper half of the stream with the corresponding points marked in the Figs.1a and 1b by the same letters, and at present it is unimportant whether the points e and f corresponding in Fig.1a to the uniform flows with $V=V_\infty$ and $\vartheta=\mp\sigma/2$, are at a finite or infinite distance from the tip of the wedge.

Referring the density of the gas ρ to its critical value and writing $K=1/(\rho_\infty V_\infty)$, we introduce, as usual, the stream function ψ and potential φ using the following differential equations and conditions:

$$d\psi=K\rho\,(u\,dy-v\,dx),\quad d\varphi=u\,dx+v\,dy$$
$$\psi(0,0)=\varphi(0,0)=0 \tag{1.1}$$

By virtue of (1.1) we have, for the chosen constant K and the linear scale, $\psi=0$ at the wedge and the axis of the flow, and $\psi=1$ at its boundary. Passing to V and ϑ for use as independent variables, we obtain the following system of Chaplygin equations /6/ for

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determining ψ and φ:

$$\varphi_\vartheta = \frac{V}{K\rho}\psi_V, \quad \varphi_V = -\frac{M^2-1}{K\rho V}\psi_\vartheta \tag{1.2}$$

or, after eliminating φ, the following single equation for ψ :

$$\begin{aligned}\rho V\,(V\psi_V\,/\,\rho)_V + (1-M^2)\,\psi_{\vartheta\vartheta} \equiv \\ V^2\psi_{VV}\,V + (1+M^2)\,\psi_V V + (1-M^2)\,\psi_{\vartheta\vartheta} = 0\end{aligned} \tag{1.3}$$

Here and in (1.2), ρ and M^2 are known functions of V, determined from the isoenergetic and isoentropic integrals and equation of state $i = i\,(\rho, s)$ in which i and s are the specific enthalpy and entropy of the gas.

As in other problems of a similar type (see e.g. /1-3, 6-8/, the boundary value problem obtained from ψ admits of a separation of variables. Also, in order to satisfy the conditions $\psi\,(V, \mp\sigma\,/\,2) = 0$, the function $\psi_\infty\,(\vartheta) \equiv \psi\,(V_\infty, \vartheta)$ is first additionally defined for all ϑ, and an even step function of period 2σ, equal to ± 1 is taken as $\psi_\infty\,(\vartheta)$. Taking this into account, we obtain the following expression for $\psi\,(V, \vartheta)$ satisfying (1.3) and the boundary value problem formulated above:

$$\psi(V, \vartheta) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\psi_k(V)\cos\nu_k\vartheta \tag{1.4}$$

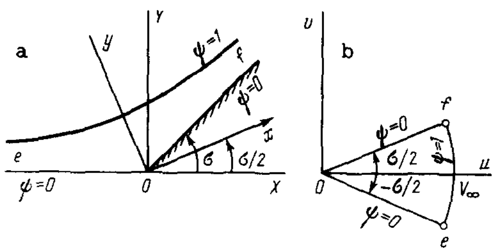


Fig.1

with $\nu_k = \pi\,(2k+1)\,/\,\sigma$ and the functions $\psi_k\,(V)$ representing the solutions of the following equation (a prime denotes a derivative with respect to V)

$$\begin{aligned}\rho V\,(V\psi_k'\,/\,\rho)' - \nu_k^2\,(1-M^2)\,\psi_k \equiv \\ V^2\psi_k'' + (1+M^2)\,V\,\psi_k' - \nu_k^2\,(1-M^2)\,\psi_k = 0\end{aligned} \tag{1.5}$$

with boundary conditions $\psi_k\,(0) = 0$ and $\psi_k\,(V_\infty) = 1$. Using a method known from the theory of ordinary differential equations /9/, we can show that the solution of (1.5) satisfying at the regular singular point $V = 0$ the condition $\psi_k\,(0) = 0$, has in its neighbourhood the form $\psi_k\,(V) = a_k V^{\nu_k}\Psi_k\,(V)$. The constant $a_k \neq 0$ is determined by the condition at $V = V_\infty$, $\Psi_k\,(V)$ is a known function of V, and $\Psi_k\,(0) = 1$. The uniform convergence of (1.4) is proved just as in /2, 8/, whose authors transferred to an arbitrary barotropic medium the analogous proof of Chaplygin /7/ for a real gas. Taking the latter case into account, we can easily find $\varphi\,(V, \vartheta)$ and show that the flow is symmetric about the y axis.

We shall begin by stating that by virtue of (1.4) ψ is an even function of ϑ, and therefore $\psi_\vartheta\,(V, 0) = 0$, and according to the second equation of (1.2), $\varphi_V\,(V, 0) = 0$. Since $V = 0$ at the origin of coordinates and all lines of the level ϑ, converges in it, we find, taking into account the last condition of (1.1), that $\varphi\,(V, 0) = 0$. Integrating the first equation of (1.2), with ψ_V taken from (1.4) and having found the arbitrary function of V appearing as the result from the equation just obtained, we arrive at the expression

$$\varphi(V, \vartheta) = \frac{4V}{K\pi\rho}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\nu_k}\psi_k'(V)\sin\nu_k\vartheta \tag{1.6}$$

Thus $\varphi(V,\vartheta)$ is an odd function of ϑ. Repeating the procedures of /1, 7/ and using the generalizations to an arbitrary gas carried out in /2, 8/, we can show that the series obtained on the right-hand side of (1.6) diverges when $\vartheta = \mp\sigma/2$, for $V_\infty < 1$, and converges for $V_\infty = 1$, i.e. when the flow is sonic. The latter means that the SL, which in this case the boundary of non-uniform subsonic flow adjacent to the tip of the wedge, are situated at a finite distance from the tip.

The symmetry of the flow which follows, in fact, from the properties of $\psi(V,\vartheta)$ and $\varphi(V,\vartheta)$ as functions of ϑ, is proved as follows. By virtue of (1.1) we have

$$dx = \frac{\cos\vartheta}{V}\,d\varphi - \frac{\sin\vartheta}{K\rho V}\,d\psi, \quad dy = \frac{\sin\vartheta}{V}\,d\varphi + \frac{\cos\vartheta}{K\rho V}\,d\psi \tag{1.7}$$

Incidentally, from the expression for dx we see that the line $\vartheta = 0$ on which, as has already been shown $\varphi = 0$, coincides with the y axis. From (1.7), taking into account (1.2) we obtain the following expressions along the isolines $\vartheta = \text{const}$:

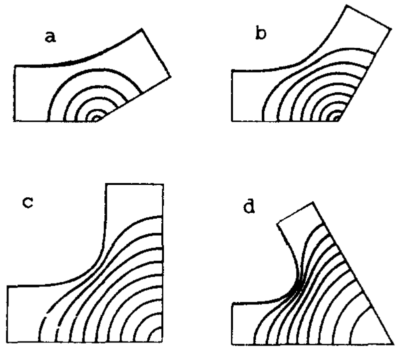
$$\begin{aligned} Kdx &= \left[\frac{M^2-1}{\rho V^2}(\cos\vartheta)\,\psi_\vartheta - \frac{\sin\vartheta}{\rho V}\,\psi_V\right]dV \\ Kdy &= \left[\frac{M^2-1}{\rho V^2}(\sin\vartheta)\,\psi_\vartheta + \frac{\cos\vartheta}{\rho V}\,\psi_V\right]dV \end{aligned} \tag{1.8}$$

Since according to (1.4) both ψ_V, and ψ, are even functions and ψ_ϑ is an odd function of ϑ, it follows that by virtue of (1.8) the lines $\vartheta = \pm\theta$ with $0 \leqslant \theta \leqslant \sigma/2$ are symmetrical about the y axis, and for any fixed $V < V_\infty$ we have on them identical y of the same sign and x of different sign. This means that the lines of constant V, or of any other scalar parameter related to V, are symmetrical about the y axis which is a bisectrix of the outer angle between the axis and the generator of the wedge. Lastly, the rectilinearity of the SL follows, in particular, from (1.7), provided that we take into account the fact that when $V_\infty = 1$, then, as we have already remarked, $\mp\varphi(1, \mp\sigma/2) = \text{const} < \infty$. Therefore, we have on these lines $d\varphi = 0$, and from (1.7) we have $dy/dx = -\operatorname{tg}(\mp\sigma/2) = \pm\operatorname{tg}(\sigma/2)$, where the upper (lower) sign corresponds to the left (right) SL. Therefore the SL are rectilinear normal axes of the flow and of the geneator of the wedge.

2. The interactions of the sonic flow with wedge-shaped bodies were calculated for a real gas with $\varkappa = 1.4$, for which the functions $\psi_k(V)$ and $\psi_k'(V)$ in (1.5) and (1.6) were given by the solutions of the hypergeometric equation and in terms of Bessel functions. Computations have shown that in order to construct a sufficiently accurate solution, and in particular to determine the boundaries of the flow, the SL and the isolines $V = \text{const}$, especially for values of V close to unity, it is necessary to employ a very large number (over 100) of terms of the expansions (1.4), (1.6) and similar. The series are, as a rule, not of constant sign, and their sums may be smaller than their first terms by many (up to 15) orders of magnitude, which imposes very stringent demands on the accuracy of the computations. It was to overcome such difficulties that a wider range of examples using the well-known asymptotic representation of the functions needed were compiled /10-12/.

A program which was written taking the above difficulties into account, makes it possible to obtain the necessary characteristics of the flows in question in a reasonably short time. Thus, a computation yielding 10 isolines $V = \text{const}$ takes, for $\sigma = 30, 60, 90$ and 120°C, 10-30 sec on the ES-1061 computer. An additional check of the accuracy of the results can be carried out by comparing the force acting on the initial "subsonic" segment of the wedge obtained by integrating p over its surface, with the exact value obtained from the integral law of conservation of momentum /1/. In the calculations the error did not exceed 2.5%.

Fig.2 and 3 show some results of the computations. In Fig.2a-d we have the isolines (including the boundaries of the flow and the SL for the angles $\sigma = 30, 60, 90$ and 120° using the step $\Delta V = 0.1$. As we know, the solution for an incompressible fluid holds near the stagnation point, and the solution implies that $V \sim r^n$, where r is the distance to the tip of the wedge and the positive exponent n is smaller than, equal to or greater than unity for $\sigma < 90°$, $\sigma = 90°$ and $\sigma > 90°$ respectively. Therefore, in Fig.2a and b the isolines bunch closer together towards the tip of the wedge, and even coalesce with it within the limits of the reproduction, while in Fig.2c at $V \leqslant 0.6$ they are almost circular and separated by practically identical increments in the value of r. Finally, Fig.3 shows the dependence of l/h, on σ, where l is the distance between the tip of the wedge and any SL, and the initial segment of the curve is shown on a larger scale (the left-hand scale). In the ideal-gas approximation the solution for a wedge of finite length $L \geqslant l$ does not differ from the solution already constructed, and when the sonic flows leave the wedge, they make angles $\pm\sigma$, with the X axis while the region between them contains gas at rest, at the critical pressure.



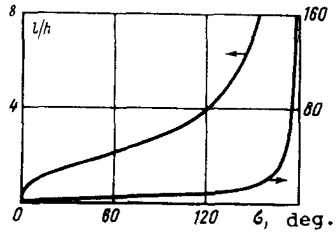


Fig.2 Fig.3

We will make three remarks. Firstly, the properties of a flow in which a homogeneous stream interacts with a wedge described above, can be extended as a corollary with $M_\infty = \text{const} \leqslant 1$, to embrace the inhomogeneous (non-isoenergetic and non-isentropic) flows of real gas with constant p_∞ and p_*. This follows from the results of /13/, (see also /14/), which enable us to reduce the problem of an inhomogeneous flow impinging on a wedge, to the problem discussed above, by referring, for each ψ, the velocity and density of such flows to their ψ-dependent critical values V_*° and ρ_*°, and the pressure to the constant (in this case) product $\rho_*^\circ (V_*^\circ)^2$. Secondly, the property of the boundedness of the region G at $V_\infty = 1$ and $V \neq 1$ by the rectilinear SL is also retained for the bodies differing, in some neighbourhood of the tip from the shape of the wedge, e.g. by having rounded tips. Naturally, the Fourier method in hodograph variables cannot be used here, but by virtue of the theorem of /4/ the SL forming the boundary of G from left and right are rectilinear, normal to the axes of the flow and of the wedge generator. An analogous situation still occurs when the flow past an altered initial segment of the body is accompanied by the appearance of local supersonic waves. Here the only important aspect of this is that such zones are not adjacent to the left or right boundary of G. The increase in entropy at the discontinuities enclosing them can be neglected. Thirdly and finally, by virtue of the same theorem of /4/ which holds also in the axisymmetric case, and of the results of /15, 16/, the left SL which forms one of the boundaries of the region G with $V \neq 1$, is also rectilinear and normal to the axis of the flow in the case of a symmetrical sonic flow impinging on a cone-shaped body. Of course, in this case G is bounded only from above along the stream. In the downstream direction the region with $V < 1$ extends without limit (narrowing gradually to zero as $x \to \infty$).

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ANALYSIS OF THE PARADOX OF THE INTERACTION OF A VORTEX FILAMENT WITH A PLANE*

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It will be shown that the crisis occurring in the interaction of a vortex filament with a plane perpendicular to it, consisting of the failure of the solution to exist at a finite Reynolds number, is due to the formation of an extremely intense induced jet in the axial region. External and internal expansions are constructed for the near-critical situation, yielding an exhaustive characterization of the structure of the solution. It is observed that if the circulation is specified not on the axis but on a finite-angled cone, the solution continues to exist at all Reynolds numbers. The vortex filament is considered as a limiting case of a cone.

The interaction of a vortex filament with a plane has been studied in many publications, beginning with /1/, where a paradoxical fact was established: a solution with bounded meridian velocity exists at Reynolds numbers not exceeding a certain critical value, ceasing to exist at higher Reynolds numbers. A detailed exposition of these results and a survey of the literature up to 1980 can be found in /2/. As shown in this paper, the source of the paradox is that the axis of symmetry lies in the flow region. The motion under consideration may be interpreted as being generated by a rotating needle at right angles to the plane. In this case, however, since the dimensions of the needle are finite, its immediate vicinity is a region of non-selfsimilarity, which remains outside the scope of the discussion.

Serrin /3/ assumed that the longitudinal component of the velocity has a logarithmic singularity along the axis, treating the coefficient of the logarithm as a parameter in addition to the circulation. Serrin showed that the plane of these parameters contains a curve which bounds the region of existence of the solutions of the class in question. The value of the coefficient of the logarithm was determined by postulating an additional hypothesis of a phenomenological nature.

In this paper a different approach is taken. The flow nucleus is situated in a small-angled cone, on whose surface the circulation and other appropriate boundary conditions are given. The angle is then allowed to approach zero. The longitudinal component of the velocity of the external flow remains bounded, but in the critical situation a singularity forms along the axis - a linear sink of well-defined strength. At supercritical Reynolds numbers the passage to the limit of a vortex filament produces the same external flow, identical with the critical flow.

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